Physics 281

Lecture II → A Second Piece of Pi
→ Turbulence

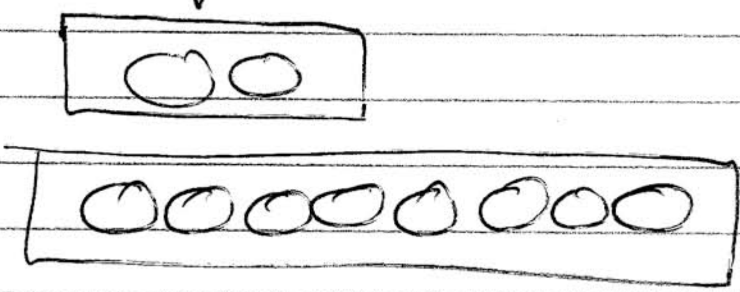
→ trickier example:

~ How does speed scull scale with
# rowers?

ℓ

i.e. pair

eight



(bireme vs trireme)

Some basic ideas:

~ racing sculls ride on surface
⇒ drag set by skin friction
(assume)

~ n rowers → ① dimension (length boat)
Power/oarsman ~ A ~ const.

$G \sim \ell^3/n$ ~ volume/oarsman ~ const n.b.: 2 'length' scales here:

$G^{1/3}$ → length scale / oarsman
n → # rowers.

$F_d \sim \rho v^2 l^2$, as usual
ρ ← speed → length

$P \sim v F_d \sim \rho v^3 l^2 \sim A n$

power required — total crew power

and have $l \sim (nG)^{1/3}$.

so parameters: $(A = P/n)$

$V, \rho, P, \underbrace{G, n}_{2 \text{ scales}}$

dimensions:

L, T, M, n

$\Rightarrow$ 1 Π variable.

$$\Pi = \frac{P}{An} \sim \frac{\rho V^3 \ell^2}{An} \sim \text{const.}$$

$$\sim O(1) \quad \text{, obvious}$$

$$\ell \sim (nG)^{1/3}$$

$$\Pi \sim \frac{\rho V^3 (nG)^{2/3}}{A\, n} \sim O(1)$$

$$\Rightarrow \quad V \sim (An)^{1/3} / \rho^{1/3} (nG)^{2/9}$$

$$\boxed{V \sim n^{1/9} A^{1/3} / \rho^{1/3} G^{2/9}}$$

$V \sim n^{1/9} A^{1/3} / \rho^{1/3} G^{2/9}$ $\rightarrow$ speed scales as $n^{1/9}$!

$\Rightarrow$ Admiral Arius should feed crew better, rather than increasing # rowers.

Some observations:

→ in Π theory quantities constructed multiplicatively.

obviously Π thm. inapplicable to additive quantities

→ scaling → power law

i.e. $V \sim n^{1/9} A^{1/3} / \rho^{1/3} \sigma^{2/9}$

rescaling indep. dimensional quantity ⇒ (multiplicative factor)(#) * (original)

Prove: It is always possible to pass from chosen original system of units to some other system within same class, such that any quantity, say a_1, in the set of quantities with independent dimensions $a_1, \ldots a_k$ changes its numerical value by specified factor A_1 while other quantities unchanged.

Proof:

→ systems of units $P, Q, \ldots$

i.e. L, T etc.

→ old $a_1, \ldots a_k$ → indep. param (dim.)

new $a_1', \ldots a_k'$

so

$$a_1' = a_1 P^{\alpha_1} Q^{\beta_1} \ldots$$

$$a_2' = a_2 P^{\alpha_2} Q^{\beta_2} \ldots$$

$$\vdots$$

$$a_k' = a_k P^{\alpha_k} Q^{\beta_k} \ldots$$

and seek:

$$a_1' = A_1 a_1 \qquad (\#\text{ factor})$$

$$a_2' = a_2$$

$$\vdots$$

$$a_k' = a_k$$

∴ have system eqns:

$$P^{\alpha_1} Q^{\beta_1} \ldots = A_1$$

$$P^{\alpha_2} Q^{\beta_2} \ldots = 1$$

$$P^{\alpha_K} Q^{\beta_K} \ldots = 1$$

take lns:

$$\ln A_1 = \alpha_1 \ln P + \beta_1 \ln Q + \ldots$$

$$0 = \alpha_2 \ln P + \beta_2 \ln Q + \ldots$$

$$0 = \alpha_K \ln P + \beta_K \ln Q + \ldots$$

⇒ system has at least one solution.

Check: Fails if:

$$\alpha_1 \ln P + \beta_1 \ln Q + \ldots = c_2 (\alpha_2 \ln P + \beta_2 \ln Q + \ldots) \ldots + c_K (\alpha_K \ln P + \beta_K \ln Q + \ldots)$$

i.e. RHS 1 = Lin comb other RHS

$$\Rightarrow \quad P^{\alpha_1} Q^{\beta_1} \cdots = \left(P^{\alpha_2} Q^{\beta_2} \cdots\right)^{c_2} \cdots \left(P^{\alpha_k} Q^{\beta_k} \cdots\right)^{c_k}$$

$$\Rightarrow \quad q_1 = q_2^{c_2} \cdots q_k^{c_k}$$

contradicts indep. dimensions.

But: → same can apply to Π variables ...

→ to satisfy, must have <u>multiplicative</u> relation

$$\Pi_i = F(\Pi_1, \Pi_2 \ldots \Pi_{i-1}, \Pi_{i+1} \ldots \Pi_{n-r})$$
$$= \Pi_1^{\alpha} \Pi_2^{\beta} (\Pi_{i-1})^{c} \cdots \Pi_{n-r}^{\Omega}$$

~~must have same property.~~

⇒ effectively proves Π thm.

See Barenblatt for more details.

→ Turbulence

→ a crucial example in scaling and self-similarity is turbulence

→ self-similarity:

~ phenomenon 'looks the same' over a range of scales ℓ

$$\ell_i < \ell < \ell_o$$

ℓ_i ↓ inner, ℓ_o ↓ outer

~ 'looks same':

$$\rho(r,t) = \rho(r/R(t))$$

$$R(t) = \sigma t^{2/5}$$

$$t \to \alpha t$$

so $r \to \alpha^{2/5} r$ leaves ρ invariant

~ power law dependence is symptom

i.e. $\delta V \sim \epsilon^{1/3} \ell^{1/3}$

$$\ell \to \alpha \ell \iff \delta V \to \alpha^{1/3} \delta V.$$

Some examples:

① cascade: hierarchical fragmentation — "shattering" → 3D fluid turbulence

② aggregation ("inverse cascade")
→ colloidal aggregation, ala Schmoluchowski

③ Fractals and β-model:
→ meaning of dimension
→ fractals

① Fluid Turbulence (c.f.: Frisch)

What is it?

— spatio-temporal 'disorder'

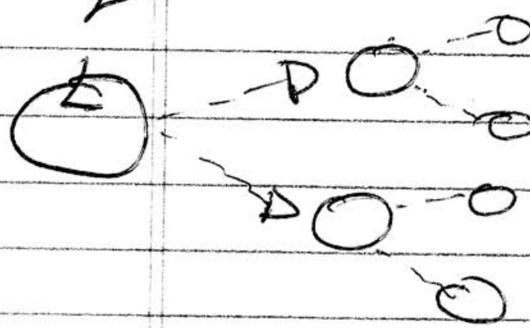
— broad range of space-time scales

— (*) power transfer thru broad range scales

— (*) energy dissipation

- can 'view' as consisting of sequence of basic interactions

i.e.

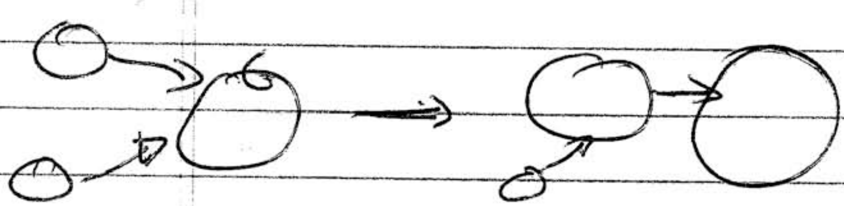


cascade

→ fragmentation sequence

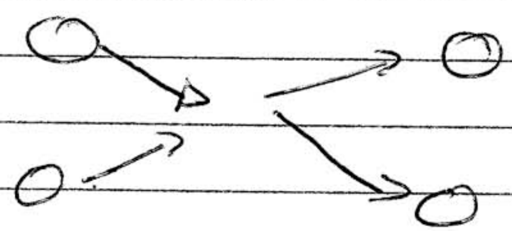
→ # eddys increase.

vs. aggregation / inverse cascade



aggregates decreases
size increases

vs. plain vanilla collision



particles conserved.

More characteristics:

- decay of large scales
- irreversible mixing
- can be intermittent / bursty.

Key parameter: $Re = v(L)\,L/\nu$

Reynolds #

$l_o / l_i \sim Re^{\alpha}$ $\quad \alpha = 3/4$

For atmospheric turbulence: BL on hot day

$Re \sim 10^8$

$l_{out} \sim$ few km

$l_{in} \sim$ few mm

Laws (Empirical)

- recall

$F_d \sim C_D A \rho V^2$

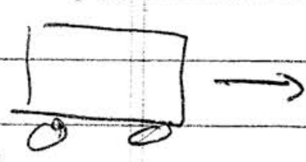
$C_D = C_D(Re)$ <u>Flat</u> in turbulent regime

$\Rightarrow$

- Finite Energy Dissipation Rate

If, in experiment on turbulent flow all control parameters kept the same except viscosity, which is lowered as much as possible, energy dissipation per mass dE/dt approaches a <u>finite</u> limit

Simple Terms: Energy dissipation is due to viscosity yet does not depend explicitly on it!



recall $F_d \sim C_D \rho S_A u^2$

$dE/dt \sim F_d u \sim \underset{\text{const}}{C_D} \rho S_A u^3$

$dE/dt \sim u^3/l \; \ell_n$

so $$\boxed{\frac{d\mathcal{E}}{dt} \sim u^3/\ell} \equiv \epsilon$$

macroscopic
length scale

Where does energy go?

$\Rightarrow$ viscous dissipation!

i.e. imagine large scale forcing

advection $\to$ no net effect $\qquad \nabla \cdot \underline{v} = 0$

$$\partial_t \langle \tilde{v}^2 \rangle + \langle \nabla \cdot (\underline{v}\,\underline{v}) \rangle = -\nu \langle (\nabla \underline{v})^2 \rangle$$

$$- \langle \nabla \cdot (\underline{v} p) \rangle + \langle \underline{f} \cdot \underline{v} \rangle$$

pressure - no net effect

st $\Rightarrow \quad \nu \langle (\nabla \underline{v})^2 \rangle = \langle f \cdot \underline{v} \rangle$

Now, necessarily $\quad \langle \underline{f} \cdot \underline{v} \rangle = \epsilon$

so

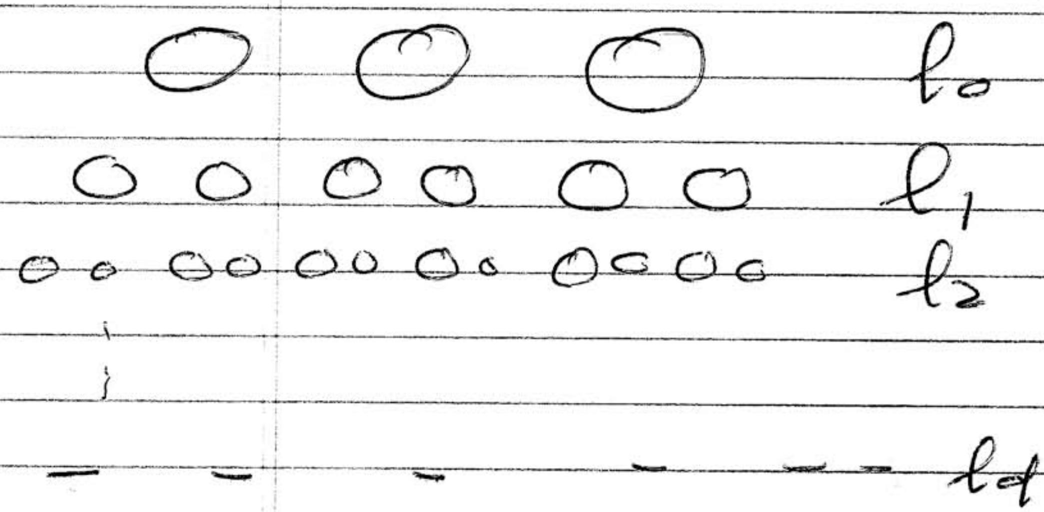
$\epsilon = \nu \langle (\nabla v)^2 \rangle \rightarrow$ balance

$\downarrow$
indep ν $\Rightarrow$ $(\nabla v)_{rms} \sim 1/\nu^{1/2}$

$\Rightarrow$ turbulence forms singular velocity gradients

$\Rightarrow$ must necessarily access small scales.

How: Cascade $\rightarrow$ hierarchical fragmentation



~ again empirical $\Rightarrow$ broad range of scales, with no gaps

How described? → structure functions!

$$\delta v(\ell) = \left((\underline{v}(\underline{r}+\underline{\ell}) - \underline{v}(\underline{r})) \cdot \frac{\underline{\ell}}{|\ell|} \right)$$

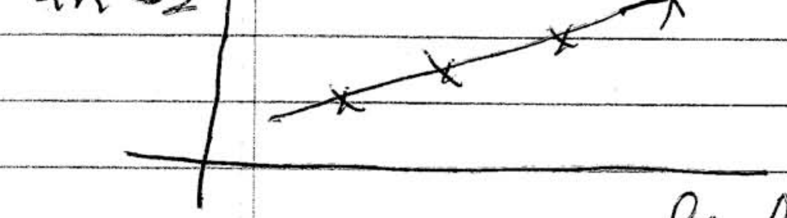
difference in velocity accross scale ℓ

$$\Rightarrow \langle \delta v(\ell)^2 \rangle \quad \ldots \quad \langle \delta v(\ell)^n \rangle$$

related energy distribution in scale

⇒ 2/3 Law (Empirical)

$$S_2(\ell) = \langle (\delta v(\ell))^2 \rangle \sim \ell^{2/3}$$



→ Rigorous: $\boxed{\langle (\delta v(\ell))^3 \rangle = -\frac{4}{5} \epsilon \ell}$

4/5 Law.

~ energetics

→ What's the story?

– K41 (Kolmogorov Phenomenology)

Ideas:

– Flux of energy in scale space from l_0 (input/integral scale) to l_d (dissipation scale – set by ν).

– energy flux is same at all scales between l_0, l_d $\Leftrightarrow$ self-similarity

– energy dissipation – set as $\nu \to 0$ but not $= 0$

– symmetry breaking of stirring, etc. lost $\Rightarrow$ symmetry restored.

Ingredients / Players

→ exciton → eddy

→ l: scale parameter, eddy scale

$\Rightarrow v(\ell)$ $\qquad v(\ell) \sim \langle \delta v_{\parallel}(\ell)^2 \rangle^{1/2}$

$$\delta v_{\parallel} \sim \left[v(\underline{r}+\underline{\ell}) - v(\underline{r}) \right] \cdot \frac{\underline{\ell}}{\ell}$$

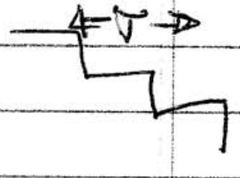
velocity increment on ℓ

$\Rightarrow V_0$: rms eddy fluctuation (large scale dominates)

$$V(\ell_0) \sim V_0$$

$\Rightarrow \tau(\ell)$: eddy transfer / life time / turn-over rate

$\Rightarrow$ characteristic scale of transfer in cascade step



Now, self-similarity $\Rightarrow$ constant flow-thru rate:

$$\epsilon = v(\ell)^2 / \tau(\ell).$$

$\tau(\ell)$?

$T(\ell)$:

- dimensionally → 'lifetime' of structure of scale ℓ
  → time to distort out of existance

For scale ℓ, which ℓ' affect lifetime?

- $\ell' \gg \ell$ (?)

advect eddy, but don't distort it.

⇒ irrelevent - physics not change under random Galilean boost

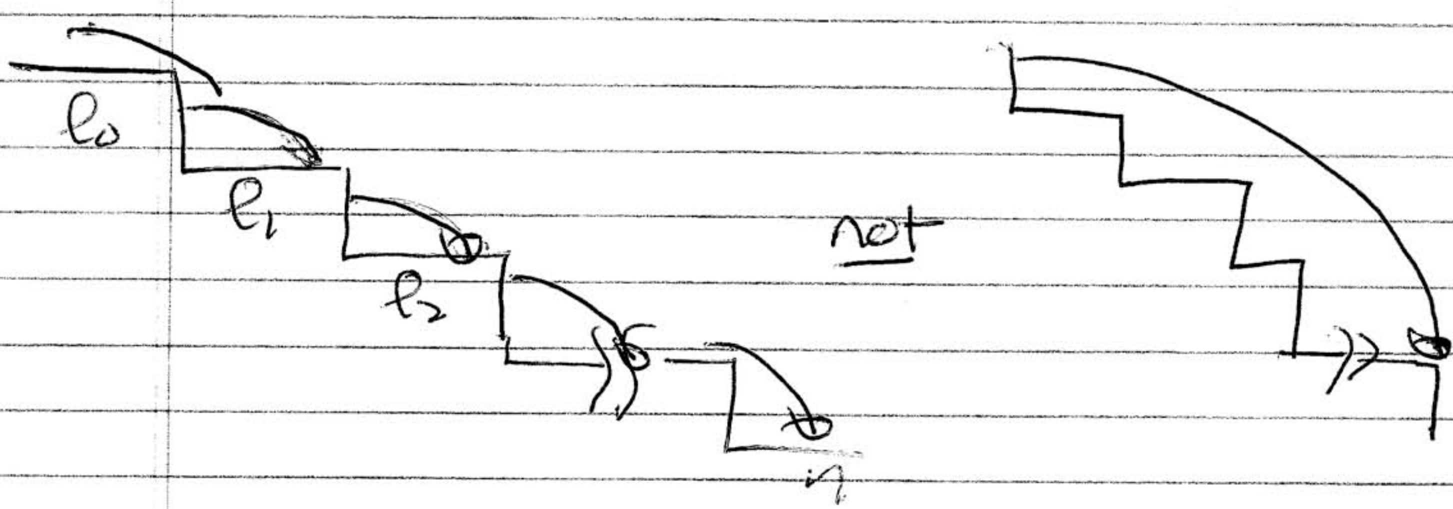
⇒ violates symmetry restoration

- scales $\ell' \ll \ell$
  ~ irrelevent, as very little energy / shear in such eddys

$\Rightarrow$

- strongest interaction on $\ell' \sim \ell$. Comparable scales distort one another

cascade is local in scale:



so $\quad \tau(\ell) \sim \ell / v(\ell)$

$$\Rightarrow \quad \epsilon \sim \frac{v(\ell)^2}{\tau(\ell)} \sim \frac{v(\ell)^3}{\ell}$$

i.e.

$$\frac{v_0^3}{\ell_0} \sim \frac{v(\ell)^3}{\ell}$$

$$\boxed{v(\ell) \sim (\epsilon \ell)^{1/3}}$$

$$\boxed{\begin{aligned} v(\ell)^2 &\sim \epsilon^{2/3}\,\ell^{2/3} \\ &\sim v_0^2\,(\ell/\ell_0)^{2/3} \end{aligned}}$$

- Power law
- recovers '2/3 law'
- dependence on ℓ_0, v_0 only via ϵ

For k spectrum:

if $\quad E(k) = |v(k)|^2$

s/t $\quad E = \int dk\, |v(k)|^2 = \int dk\, E(k)$

i.e. absorb density of states

then

$$v(\ell)^2 = \int_{k_{\ell_{n-1}}}^{k_{\ell_n}} dk\, E(k)$$

$$v(\ell)^2 \sim \epsilon^{2/3} \ell^{2/3} \sim \epsilon^{2/3} k_\ell^{-2/3}$$

$$\boxed{E(k) \sim \epsilon^{2/3} k^{-5/3}}$$

Kolmogorov spectrum.

N.B.: $1/\tau(\ell) \sim v(\ell)/\ell \sim \epsilon^{1/3}/\ell^{2/3}$

transfer rate increases as scale decreases.

finite time to end!

i.e. total time:

$$T = \sum_{n=0}^{\infty} \tau_n$$

$$= \sum_{n=0}^{\infty} \frac{\ell_0}{v_0} \left(\frac{\ell_n}{\ell_0}\right)^{2/3}$$

$\ell_n/\ell_0 \sim \alpha^n$ $\qquad\qquad$ $\alpha < 1$

$$T = \sum_{n=0}^{\infty} \frac{\ell_0}{v_0} \alpha^{2n/3}$$

$$T \sim \frac{l_0}{v_0} \frac{1}{1-\alpha^{2/3}}$$

$\Rightarrow$

- T_0 sets cascade time
- cascade can go thru ∞ # steps in finite time
- hence analogy with "shattering"

$\rightarrow$ For dissipation scale l_d

- occurs when viscous diffusion kicks in and cuts-off cascade
- $1/T(l) \sim \nu/l^2$ $\rightarrow$ diffusive and NL time scales cross
- $\epsilon^{1/3}/l^{2/3} \sim \nu/l^2$

$$\boxed{l_d \sim \nu^{3/4}/\epsilon^{1/4}}$$

$\rightarrow$ dissp'n scale

$\Rightarrow$ Finally

$$\#\text{ DOF's} \sim \left(l_0/l_i\right)^3$$

$$\sim \left(l_0/l_d\right)^3$$

$$\sim \left(Re^{3/4}\right)^3 \sim Re^{9/4}$$

For $l_0 \sim 1\,km$, $l_d \sim 1\,mm$ $\Rightarrow N \sim 10^{18}$.

N.B.: What is missing?